Then, using (9) with allowance for (12) and (10), we determine $p_{a_i}{}^*$ at the same points. This is followed by the determination of functions $\zeta_i{}^j$ and $\psi_i{}^j$ for the next layer, using the new value of $p_{a_i}{}^*$. This procedure is repeated until the criterion of completion of the transient process – stabilization of the oscillation mode of the elastic system – is reached (Fig. 5). Completion of this process can be also judged by the behavior of transient functions $p_{a_i}{}^* (\tau)$ (see Fig. 6).

The obtained numerical data are shown in Figs. 3 and 4 in the form of curves $\zeta (\xi, \tau)$ and $p_a{}^* (\xi, \tau)$ for $p^* = 3$. Examination of Fig. 3 shows that the continuous change of the median surface shape with a shift of the maximum amplitude of deflection toward the plate trailing edge is a distinctive feature of deformation of the panel under transient conditions. This also follows from the diagram of distribution of parameter $p_a{}^* (\xi, \tau)$ shown in Fig. 4. A set of curves related to the points lying at distances equal $^1/_5$ and $^4/_5$ of the plate length from the plate leading edge is shown in Fig. 5. It is seen that perturbations in the stream result in rapid increase of deflection, beginning at points in the neighborhood of the trailing edge and, then, with a certain lag, at points close to the leading edge. Later, when oscillations become stabilized, this process is somewhat attenuated. Transient functions $p_{\alpha_i}{}^*(\tau)$ and $p_{ai}{}^*(\tau)$ for points of the panel to which in Fig. 5 relate curves $\zeta_i (\tau)$ are shown in Fig. 6, where the different rates of pressure increase with time are a significant feature of curves $p_{a_i}{}^* (\tau)$.

We note in conclusion that the proposed method makes it possible not only to establish the deformation pattern of the panel median plane and pressure distribution with respect to time but, also, to determine unsafe stresses in the structure in transient mode and stable oscillations.

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PLANE CONTACT PROBLEM FOR A LINEARLY-DEFORMABLE FOUNDATION IN THE PRESENCE OF ADHESION

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On the basis of a spectral relationship for the Jacobi polynomials which is more general than that used in [1], a method is proposed for solving the plane contact problem for a linearly-deformable foundation of general type whose particular case is a half-space with an elastic modulus of the form

$$E = E_\nu z^\nu \qquad (0 \leqslant \nu < 1)$$

The exact solution of the plane problem of the impression of a die with

adhesion was apparently given first for this case and it was shown that the contact stresses at the edges of the die tend to infinity as in the case of an ordinary half-space, changing sign an infinite number of times. The way is indicated for solving approximately the bending problem of a finite beam adhering to a linearly-deformable foundation. The inaccuracy in the paper [2] devoted to this same problem in application to the mentioned particular case of a foundation is pointed out.

1. The necessary information about a linearly-deformable foundation is related to giving the displacement of the surface points of the foundation due to the effect of a concentrated force. Let $\theta_0 v_0(x)$, $-\theta_3 v_4(x)$, respectively, denote the vertical and horizontal displacements of the mentioned points due to a unit vertical force applied at the point taken as the origin [3, 1]. Let $\theta_1 v_1(x)$, $\theta_2 v_2(x)$ denote analogous displacements due to a horizontal force. If the foundation under consideration is elastic, then $\theta_1 v_1(x) = \theta_3 v_3(x)$ because of the reciprocity law. Let us consider [1, 3] the influence functions introduced to be representable as follows:

$$\begin{matrix} v_{0,2}(x) \\ v_{1,3}(x) \end{matrix} = \frac{1}{\pi}\int_0^\infty \begin{pmatrix} \varphi_{0,2}(t)\cos tx \\ \varphi_{1,3}(t)\sin tx \end{pmatrix} \frac{dt}{t} \qquad (\varphi_{0,2}(0) = 0) \tag{1.1}$$

but let us assume a more general asymptotic behavior at infinity for the densities $\varphi_m(t)$, which will agree with that taken in [4]

$$\varphi_m(t) = t^\nu [1 + O(t^{-\varepsilon})] \qquad (t \to \infty,\ 0 \leqslant \nu < 1,\ \varepsilon > 0,\ m = 0, 1, 2, 3) \tag{1.2}$$

Let a die, on which an arbitrary system of forces with principal moment M (pole of reduction at the midpoint of the section) and with principal vector components P (vertical) and Q (horizontal) acts, adhere to the section $(-a, a)$ of the surface of the foundation. It is required to determine the normal $p(x)$ and tangential $q(x)$ contact stresses. If $g_1(x)$ and $g_2(x)$, respectively, denote the vertical and horizontal displacements of the contact points, then the problem can be expressed in the form of the following system of integral equations [1, 3, 4]:

$$\theta_0 \int_{-a}^{a} v_0(x-y)\,p(y)\,dy + \theta_1 \int_{-a}^{a} v_1(x-y)\,q(y)\,dy = g_1(x) \tag{1.3}$$

$$\theta_2 \int_{-a}^{a} v_2(x-y)\,q(y)\,dy - \theta_3 \int_{-a}^{a} v_3(x-y)\,p(y)\,dy = g_2(x)$$

$$(|x| < a)$$

If the die has a flat base, then

$$g_1(x) = \delta + \theta x, \quad g_2(x) = \varepsilon \tag{1.4}$$

where δ is the vertical settlement of the die, θ is the angle of rotation, and ε is the horizontal displacement of the die, whose magnitudes should be determined from the equilibrium conditions of the die.

In order to apply the method of orthogonal polynomials [5] to the solution of the system (1.3), let us isolate the irregular parts of the influence functions by using (1.1) and formula 3.761 from [6], i.e. let us represent them as

$$v_{0,2}(x) = \frac{\Gamma(\nu+1)\cos{}^1/_2\nu\pi}{\pi}\left[\frac{1}{\nu|x|^\nu} - l_{0,2}(x)\right] \tag{1.5}$$

$$v_{1,3}(x) = \frac{\Gamma(\nu)\sin{}^1/_2\nu\pi}{\pi}\left[\frac{\operatorname{sgn} x}{|x|^\nu} - l_{1,3}(x)\right]$$

$$l_{0,2}(x) = \frac{1}{\Gamma(\nu+1)\cos{}^1/_2\nu\pi}\int_0^\infty \frac{t^\nu - \varphi_{0,2}(t)}{t}\cos tx\,dt$$

$$l_{1,3}(x) = \frac{1}{\Gamma(\nu)\sin{}^1/_2\nu\pi}\int_0^\infty \frac{t^\nu - \varphi_{1,3}(t)}{t}\sin tx\,dt$$

Because of (1.2) the functions $l_m(x)$ $(m = 0, 1, 2, 3)$ will be continuous at least. Keeping (1.5) in mind, we can write the characteristic part of the system (1.3) as

$$\theta_0^* \int_{-a}^{a} \frac{p(y)\,dy}{\nu|x-y|^\nu} + \theta_1^* \int_{-a}^{a} \frac{\operatorname{sgn}(x-y)}{|x-y|^\nu} q(y)\,dy = f_1(x) \tag{1.6}$$

$$\theta_2^* \int_{-a}^{a} \frac{q(y)\,dy}{\nu|x-y|^\nu} - \theta_3^* \int_{-a}^{a} \frac{\operatorname{sgn}(x-y)}{|x-y|^\nu} p(y)\,dy = f_2(x)$$

$$(\pi\theta_{0,2}^* = \Gamma(\nu+1)\cos{}^1/_2\nu\pi\theta_{0,2}, \quad \pi\theta_{1,3}^* = \Gamma(\nu)\sin{}^1/_2\nu\pi\theta_{1,3})$$

Let us try to obtain the exact solution of this system. Let us make the substitution $x = a\xi$, $y = a\eta$, let us divide the first equation by θ_1^* and the second by θ_3^* and let us introduce the notation

$$\varkappa_*^2 = \theta_0^*\theta_2^* / \theta_1^*\theta_3^*, \qquad \varkappa^2 = \theta_0^*\theta_3^* / \theta_1^*\theta_2^* \tag{1.7}$$

$$r(\eta) = \varkappa^{1/2} a^{1-\nu} p(a\eta), \qquad s(\eta) = \varkappa^{-1/2} a^{1-\nu} q(a\eta)$$

Multiplying the first equation of the system obtained by $i\varkappa^{-1/2}$ and the second by $-\varkappa^{1/2}$ and adding the results, we obtain

$$\int_{-1}^{1} \frac{\operatorname{sgn}(\xi-\eta) + i\operatorname{ctg}{}^1/_2\lambda\pi}{|\xi-\eta|^\nu}\chi(\eta)\,d\eta = f(\xi) \tag{1.8}$$

$$(|\xi| \leqslant 1, \quad \operatorname{ctg}{}^1/_2\lambda\pi = (\theta_0\theta_2)^{1/2}(\theta_1\theta_3)^{1/2}\operatorname{ctg}{}^1/_2\nu\pi)$$

Here

$$\chi(\eta) = r(\eta) + is(\eta), \quad f(\xi) = i\varkappa^{-1/2}(\theta_1^*)^{-1} f_1(a\xi) - \varkappa^{1/2}(\theta_3^*)^{-1} f_2(a\xi) \tag{1.9}$$

The integral equation (1.8) can be solved exactly. For example, its solution in quadratures can be found in [4]. Such a form of the solution can turn out to be useful to obtain an approximate solution of the initial system (1.3) by the Carleman method of regularization. This method has been used in [3, 7] in application to a foundation in the form of an elastic layer. For the purposes of this paper, a form of the solution as a series of Jacobi polynomials $P_m^{\alpha,\beta}(z)$ is more preferable

$$\chi(\xi) = \sum_{m=0}^{\infty} \frac{f_m P_m^{\rho}(\xi)}{i\sigma_\nu\lambda_m\psi_\rho(\xi)}, \qquad P_m^{\rho}(\xi) = P_m^{-\omega-i\rho,\,-\omega+i\rho}(\xi)$$

$$\psi_\rho(\xi) = \frac{(1-\xi)^{i\rho+\omega}}{(1+\xi)^{i\rho-\omega}}, \quad \omega = \frac{1-\nu}{2}, \quad \rho = \frac{1}{2\pi}\ln\frac{\sin {}^1/_2(\nu+\lambda)\pi}{\sin {}^1/_2(\nu-\lambda)\pi} \qquad (1.10)$$

$$\sigma_\nu = \frac{2\pi[\sin {}^1/_2(\nu+\lambda)\pi \sin {}^1/_2(\nu-\lambda)\pi]^{1/2}}{\sin\nu\pi\sin {}^1/_2\lambda\pi}, \quad f_m = \int_{-1}^{1}\frac{f(\xi)P_m^{-\rho}(\xi)\,d\xi}{\psi_{-\rho}(\xi)}$$

$$\lambda_m = 2^\nu|\Gamma(1-\omega+i\rho+m)|^2[m!^2(\nu+2m)\Gamma(\nu)]^{-1}$$

Formula (1.10) results from the spectral relationship contained in [4, 8]

$$\int_{-1}^{1}\frac{[\operatorname{sgn}(\xi-\eta)+i\operatorname{ctg}{}^1/_2\lambda\pi]\,P_m^{\rho}(\eta)\,d\eta}{|\xi-\eta|^\nu\psi_\rho(\eta)} = \frac{i(\nu)_m\sigma_\nu}{m!}P_m^{-\rho}(\xi) \qquad (1.11)$$

2. The spectral relationship (1.11) permits application of the method of orthogonal polynomials to obtain an approximate solution of the system (1.3). Let us first represent it as one integral equation in $\chi(\xi)$ defined by (1.7) and (1.9). To do this, the same manipulations should be performed as had been done above with the system (1.6) taking the representation (1.5) into account here. Consequently, we will have in place of (1.3)

$$\int_{-1}^{1}\left\{\left[\frac{\operatorname{sgn}(\xi-\eta)+i\operatorname{ctg}{}^1/_2\lambda\pi}{|\xi-\eta|^\nu}-l^+(\xi-\eta)\right]\chi(\eta)-l^-(\xi-\eta)\overline{\chi(\eta)}\right\}d\eta = g(\xi)$$

$$g(\xi) = i\varkappa^{-1/2}(\theta^*_1)^{-1}g_1(a\xi) - \varkappa^{1/2}(\theta_3^*)^{-1}g_2(a\xi) \qquad (2.1)$$

$$2a^{-\nu}l^\pm(x) = l_3(ax) \pm i\varkappa_* l_2(ax) \pm l_1(ax) + i\varkappa_* l_0(ax)$$

Now, if we seek the solution of the obtained integral equation in the form of a series similar to (1.10)

$$\chi(\xi) = \sum_{m=0}^{\infty}\frac{z_m P_m^{\rho}(\xi)}{\psi_\rho(\xi)}, \qquad \overline{\chi(\xi)} = \sum_{m=0}^{\infty}\frac{\overline{z_m}P_m^{-\rho}(\xi)}{\psi_{-\rho}(\xi)} \qquad (2.2)$$

then by using the procedures of the method of orthogonal polynomials [5], an infinite system of algebraic equations in z_m is easily obtained. For example, if the regular parts of the kernel functions are approximated by polynomials of the form

$$l_m(s) = \sum_{j=0}^{n}a_j^{(m)}s^{2j} \quad (m=0,\,2), \qquad l_m(s) = \sum_{j=0}^{n}a_j^{(m)}s^{2j+1} \quad (m=1,\,3) \qquad (2.3)$$

then the mentioned infinite system of algebraic equations degenerates into a finite system

$$z_l = \frac{1}{i\sigma_\nu\lambda_l}\left[g_l + \sum_{m=0}^{N-l}\left(z_m\sum_{k=m+l}^{N}C_k^+B_{mk}^{l+} + \bar z_m\sum_{k=m+l}^{N}\bar c_k B_{mk}^{l-}\right)\right] \qquad (2.4)$$

$$(l=0,\,1\ldots N,\ N=2n+1)$$

$$z_l = g_l(i\sigma_\nu\lambda_l)^{-1} \qquad (l>N)$$

Here

$$g_m = \int_{-1}^{1}\frac{P_m^{-\rho}(\xi)}{\psi_\rho(-\xi)}g(\xi)\,d\xi \qquad (\psi_{-\rho}(\xi) = \psi_\rho(-\xi)) \qquad (2.5)$$

$$2a^{-\nu-2k}C_{2k}^{\pm} = i\varkappa_*(a_k^{(0)} \pm a_k^{(2)}), \qquad 2a^{-\nu-2k-1}C_{2k+1}^{\pm} = a_k^{(3)} \pm a_k^{(1)}$$

$$B_{mk}^{l\pm} = \int_{-1}^{1}\int_{-1}^{1} \frac{P_m^{\pm\rho}(\tau)\, P_l^{-\rho}(\xi)\,(\xi-\tau)^k\, d\xi\, d\tau}{\psi_\rho(\pm\tau)\,\psi_\rho(-\xi)} = \sum_{j=l}^{k-m} (-1)^j \binom{k}{j} b_{k-j}^m(\pm\rho)\, b_j^l(-\rho)$$

$$b_j^m(\rho) = 0, \quad j < m; \quad b_j^m(\rho) = \frac{2^{\nu+j}\Gamma(1-\omega-i\rho+j)\,\Gamma(1-\omega+i\rho+m)\,(-j)_m}{m!\,\Gamma(\nu+j+n+1)}$$

$$j > m$$

The formula (compare with [1])

$$\overline{B_{mk}^{l\pm}} = (-1)^{m+k+l} B_{mk}^{l+}$$

is useful for separating the imaginary from the real parts in the system (2.4). It should be kept in mind that the right side of the integral equation is given, according to (1.4), just to the accuracy of an additive linear function of the form

$$g(\xi) = -\varepsilon^* + i\delta^* + i\theta^*\xi \qquad (2.6)$$

$$(\sqrt{\varkappa}\,\varepsilon = \theta_3^*\varepsilon^*, \quad \delta = \sqrt{\varkappa}\,\theta_1^*\delta^*, \quad a\theta = \theta_1^*\sqrt{\varkappa}\theta^*)$$

whose parameters govern die rotation (θ) and displacement of its center of gravity (ε, δ). The die equilibrium conditions should be used to seek these parameters. Moments of the form

$$J_k = \int_{-1}^{1} \xi^k \chi(\xi)\, d\xi, \qquad k = 0, 1 \qquad (2.7)$$

should be found to obtain these latter conditions. Taking (2.2) into account and using the orthogonality of the Jacobi polynomials, we find

$$J_0 = \frac{2^\nu R_\nu z_0}{\Gamma(\nu+1)}, \quad J_1 = \frac{2^{\nu+1}R_\nu}{\Gamma(\nu+2)}\left[\frac{(1+\nu)^2+4\rho^2}{4(\nu+2)} z_1 + i\rho z_0\right] \qquad (2.8)$$

$$(R_\nu = \Gamma(1-\omega+i\rho)\,\Gamma(1-\omega-i\rho))$$

If it is taken into account that

$$[P, Q] = \int_{-a}^{a} [p(x), q(x)]\, dx, \quad M = \int_{-a}^{a} x p(x)\, dx$$

and also (1.7), (1.9), (2.7) and (2.8), then the required conditions yield

$$\varkappa^{1/2}P + i\varkappa^{-1/2}Q = \frac{(2a)^\nu R_\nu}{\Gamma(\nu+1)} z_0 \qquad (2.9)$$

$$\frac{\varkappa^{1/2}\Gamma(\nu+2)\, M}{(2a)^{\nu+1} R_\nu} = \mathrm{Re}\left[\frac{(1+\nu)^2+4\rho^2}{4(\nu+2)} z_1 + i\rho z_0\right]$$

If the die has a flat base, i.e. (1.4) holds, then the right side in the integral equation (2.1) equals the function (2.6) exactly. In this case

$$g_0 = \frac{2^\nu R_\nu}{\Gamma(\nu+1)}\left(-\varepsilon^* + i\delta^* + \frac{2\rho\theta^*}{1+\nu}\right), \quad g_1 = \frac{2^{\nu-1} i\,[(1+\nu)^2+4\rho^2]\, R_\nu\theta^*}{\Gamma(\nu+3)} \qquad (2.10)$$

$$g_n = 0 \quad (n = 2, 3, 4 \ldots)$$

and therefore, in conformity with (2.4), the series (2.2) determining the solution of the integral equation (2.1) under the condition (2.3) degenerates into a finite sum with $N+1$ terms. The representation

$$z_n = (-\varepsilon^* + i\delta^*)z_n^* + \theta^* z_n^\theta \qquad (n=0,1\ldots N) \tag{2.11}$$

should hence be used to determine the parameters ε, δ, θ from the equilibrium conditions (2.9), and also for convenience in solving the system (2.4). Here z_n^* is the solution of the system (2.4) for $g_0 = 2^\nu R_\nu \Gamma^{-1}(\nu+1)$, $g_n = 0$ $(n = 1, 2, 3 \ldots)$, z_n^θ is the solution of the system (2.4) for

$$g_0 = \frac{2^{\nu+1}\rho R_\nu}{\Gamma(\nu+2)}, \quad g_1 = \frac{i2^{\nu-1}[(1+\nu)^2+4\rho^2]R_\nu}{\Gamma(\nu+3)}, \quad g_n = 0 \quad (n \geqslant 2) \tag{2.12}$$

Substituting the values z_0 and z_1 into the conditions will result, according to (2.11), in equations for ε^*, δ^*, θ^* or according to (2.6) for ε, δ, θ.

The constructions described above are carried out under the assumption that $\nu > \lambda$. However, they remain valid even for the case $\lambda > \nu$. Only here, the following spectral relationship issuing from the results in [4, 8]

$$\int_{-1}^{1} \frac{[\operatorname{sgn}(\xi-\eta) + i\operatorname{ctg}{}^1\!/_2\lambda\pi]\, P_m^\beta(\eta)\, d\eta}{|\xi-\eta|^\nu \psi_\beta(\eta)} = \frac{(\nu)_m \mu_\nu P_m^\beta(-\xi)}{(-1)^{m+1} m!} \tag{2.13}$$

$$P_m^\beta(x) = P_m^{{}^1\!/_2\nu-1-i\beta,\ {}^1\!/_2\nu+i\beta}(x), \qquad \psi_\beta(x) = (1-x)^{1-{}^1\!/_2\nu+i\beta}(1+x)^{-{}^1\!/_2\nu-i\beta}$$

$$\beta = \frac{1}{2\pi}\ln\frac{\sin{}^1\!/_2\pi(\lambda+\nu)}{\sin{}^1\!/_2\pi(\lambda-\nu)}, \qquad \mu_\nu = \frac{2\pi[\sin{}^1\!/_2\pi(\lambda+\nu)\sin{}^1\!/_2\pi(\lambda-\nu)]^{{}^1\!/_2}}{\sin\pi\nu\sin{}^1\!/_2\lambda\pi}$$

should be used in place of (1.11).

3. Let us turn to the particular case of the linearly-deformable foundation under consideration which is an elastic half-space with elastic modulus varying as the law $E = E_\nu z^\nu$. Following [9], we discern that in this case

$$v_0(x) = v_2(x) = \nu^{-1}|x|^{-\nu}, \quad v_1(x) = v_3(x) = |x|^{-\nu}\operatorname{sgn}x$$

$$\theta_0 = \frac{(1-\mu^2)\gamma C_\nu \sin{}^1\!/_2\gamma\pi}{(1+\nu)E_\nu}, \qquad \theta_2 = \frac{(1-\mu^2)(1+\nu)C_\nu\sin{}^1\!/_2\gamma\pi}{\gamma E_\nu} \tag{3.1}$$

$$\theta_3 = \theta_1 = -\frac{(1-\mu^2)C_\nu\cos{}^1\!/_2\gamma\pi}{\nu E_\nu}, \quad C_\nu = \frac{2^{\nu+1}\Gamma[{}^1\!/_2(\nu+\gamma+3)]\,\Gamma[{}^1\!/_2(\nu-\gamma+3)]}{\pi\Gamma(\nu+2)}$$

$$\gamma = \sqrt{(1+\nu)[1-\mu\nu(1-\mu)^{-1}]} \qquad (\mu \text{ is the Poisson-s ratio})$$

If we assume

$$\gamma - 1 = \lambda \qquad (\varkappa = (1+\lambda)(1+\nu)^{-1}) \tag{3.2}$$

then the contact problem under consideration for the linearly-deformable foundation (3.1) can be reduced in conformity with Sect. 1 to the integral equation (1.8) with $f(\xi) = g(\xi)$ and its exact solution can be obtained either in the form of quadratures or series (1.10). It is hence easy to show that $\nu - \lambda > 0$ always for the foundation under consideration. If a die with flat base is kept in mind, i.e. (1.4), (2.6) and (2.10) are assumed to be valid, then the equilibrium conditions (2.9) for such a die will become

$$\varkappa^{1/2}P + i\varkappa^{-1/2}Q = \frac{(2a)^\nu R_\nu}{i\Gamma(\nu+1)\sigma_\nu}\left(-\varepsilon^* + i\delta^* + \frac{2\rho\theta^*}{1+\nu}\right) \tag{3.3}$$

$$\frac{\Gamma(\nu+2)\,\sigma_\nu \varkappa^{1/2} M}{(2a)^{\nu+1} R_\nu} = -\rho\varepsilon^* + \frac{\theta^*}{1+\nu}\left[2\rho^2 + \frac{(1+\nu)^2+4\rho^2}{2\nu(\nu+2)}\right]$$

Hence, taking account of (2.6) we obtain

$$\delta = \frac{\Gamma(1+\nu)\,\sigma_\nu \varkappa \theta_1 P}{(2a)^\nu R_\nu}$$

$$\frac{\theta}{\theta_1} = \frac{\nu\sigma_\nu \Gamma(\nu+1)\,T}{R_\nu (2a)^\nu} \qquad \left(T = \frac{(\nu+2)(\nu+1)\,[a^{-1}(\nu+1)\,M + 2\rho Q\varkappa^{-1}]}{(1+\nu)^2+4\rho^2}\right) \tag{3.4}$$

$$\varepsilon = \frac{\Gamma(\nu+1)\,\sigma_\nu\theta_1}{(2a)^\nu R_\nu}\left\{\left[1 + \frac{4\nu(\nu+2)\,\rho^2}{(1+\nu)^2+4\rho^2}\right]\frac{Q}{\varkappa} + \frac{2\nu(\nu+1)(\nu+2)\,\rho M}{a\,[(1+\nu)^2+4\rho^2]}\right\}$$

Taking these formulas as well as (1.7), (1.9), (1.10), (2.6), (2.10) and (3.2) into account, after separating real and imaginary parts, we obtain the desired contact stresses under a flat die subjected to the forces P and Q and the moment M

$$p(x) = \frac{\Gamma(\nu+1)(2a)^{-\nu}}{R_\nu (a^2-x^2)^\omega}\left\{\left[P + \frac{xT}{a(\nu+1)}\right]\cos\left(\rho\ln\frac{a+x}{a-x}\right) - \left[\frac{Q}{\varkappa} - \frac{2\rho T}{\nu+1}\right]\sin\left(\rho\ln\frac{a+x}{a-x}\right)\right\} \tag{3.5}$$

$$q(x) = \frac{\Gamma(\nu+1)(2a)^{-\nu}\varkappa}{R_\nu (a^2-x^2)^\omega}\left\{\left[\frac{Q}{\varkappa} - \frac{2\rho T}{\nu+1}\right]\cos\left(\rho\ln\frac{a+x}{a-x}\right) + \left[\frac{xT}{a} + P\right]\sin\left(\rho\ln\frac{a+x}{a-x}\right)\right\}$$

If only a compressive force ($T = Q = 0$) acts on the die, then it is seen that stress pulsations at the outer contact points, noted in [10] for an ordinary half-space, will hold in the case of an inhomogeneous half-space with $E = E_\nu z^\nu$.

Letting $\nu \to 0$ in the formulas obtained, we obtain all the known formulas for an ordinary half-space. Let us note that it is convenient to use the following representation resulting from formulas 8.381 (4) and 8.384 (6) from [6]

$$\frac{\pi}{R_\nu} = \frac{2^\nu}{\Gamma(\nu)}\int_0^{\pi/2} \operatorname{ch} 2\rho t \cos^{\nu-1} t\, dt \qquad (0<\nu<1) \tag{3.6}$$

to evaluate the R_ν given by the formula from (2.8).

4. Let a beam of finite length $2a$ and height h on which act the vertical $p_1(x)$ and horizontal $q_1(x)$ loads, adhere to the linearly-deformable foundation considered in Sect. 1. It is easy to show that the problem of designing such a beam can be reduced to the solution of the following integro-differential system:

$$\theta_0 \int_{-a}^{a} v_0(x-y)\,p(y)\,dy + \theta_1 \int_{-a}^{a} v_1(x-y)\,q(y)\,dy = w(x) \tag{4.1}$$

$$\theta_2 \int_{-a}^{a} v_2(x-y)\,q(y)\,dy - \theta_3 \int_{-a}^{a} v_3(x-y)\,p(y)\,dy = u(x) + \frac{h}{2}\frac{dw}{dx}$$

$$D\frac{d^4 w}{dx^4} = p_1(x) - p(x) - \frac{h}{2}\left[\frac{dq}{dx} + \frac{dq_1}{dx}\right], \quad C\frac{d^2 u}{dx^2} = q(x) - q_1(x)$$

Here w is the beam deflection, u is the horizontal displacement, D is the beam bending stiffness, and C is the beam tensile stiffness. Support conditions at the beam endpoints must still be added to the integro-differential system written down. If the beam ends are free, then these conditions can be replaced by the equilibrium conditions for the beam as a rigid body. Inverting the differential operators in (4.1) by using a Green function [11], we can eliminate $w(x)$ and $u(x)$ from the first two equations.

As in the case of the problem of a die, we consequently obtain a system of integral equations in the contact stresses (we eliminate the derivative of $q(x)$ contained in the third equation by integration by parts). Exactly as in the case of the die, the right sides of the integral equations will contain arbitrary constants which should either be found from the support conditions for the beam ends, or from the equilibrium conditions. The mentioned system can be reduced to an integral equation of type (2.1) by the procedures described in Sect. 1, and the method elucidated in Sect. 2 can be used for its approximate solution.

It should be stated that the paper [2] is devoted to the problem considered here in application to the particular case of a linearly-deformable die. In that paper it is proposed to use a series of Gegenbauer polynomials

$$[p(\xi a),\ q(\xi a)] = \frac{1}{(1-\xi^2)^{\omega}} \sum_{m=0}^{\infty} [A_m,\ B_m]\, C_m^{1/2\nu}(\xi) \tag{4.2}$$

in place of the series (2.2) for the contact stresses for the approximate solution of the corresponding system (4.1).

The disadvantage of such a representation in contrast to (2.2) is that the true nature of the singularities in the stresses sought at the beam ends is not taken into account here. This results in the fact that in the neighborhood of the beam ends the approximate solution will differ arbitrarily from the exact solution. However, the main deffect in [2] is that, firstly, the author took $\theta_0 = \theta_2$ in addition to the normal equality $\theta_1 = \theta_3$ in the corresponding system (4.1) (evidently by analogy with an ordinary half-space); as is seen from (3.1), this latter equality does not hold. Secondly, in selecting the desired contact stresses in the form (4.2), the author of [2] relied on the established fact [8] that Gegenbauer polynomials are eigenfunctions of the series $|x-y|^{\nu}$ $(0<\nu<1)$ with which the kernels $v_0(x-y)$ and $v_2(x-y)$ of the system (4.1) agree in the case of the foundation (3.1). But, in conformity with (3.1), still another kernel

$$\operatorname{sgn}(x-y)\,|x-y|^{-\nu} \tag{4.3}$$

is contained in the system mentioned.

Referring to the spectral relationship (1.11) obtained in [4], the author of [2] asserts that Gegenbauer polynomials are also eigenfunctions even for the series (4.3). However, this assertion is inaccurate. In fact, the relationship mentioned in [8]

$$\int_{-1}^{1} \frac{\operatorname{sgn}(x-y)\, P_m^{\nu/2,\,\nu/2-1}(y)\, dy}{|x-y|^{\nu}(1-y)^{\nu/2}(1+y)^{1-\nu/2}} = \frac{\pi(\nu)_m P_m^{\nu/2-1,\,\nu/2}(x)}{\sin {}^1/_2\nu\pi m!}$$

results from the spectral relationship (1.11).

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ON THE THEORY OF ELASTIC NONHOMOGENEOUS MEDIA WITH A REGULAR STRUCTURE

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We give the theoretical description of regular elastic structures in an unbounded elastic medium with congruent (doubly-periodic) groups of arbitrary foreign inclusions. Within the limits of a group the elastic characteristics of the inclusions are distinct and their configurations are sufficiently arbitrary. We construct a model anisotropic medium which has the rigidity of the original structure. References on problems of the theory of elastic regular structures can be found in [1 – 3].